Core 4 June 2011 Worked Solutions

① (a) $f(x) = 4x^3 - 13x + 6$

$f(-2) = 4(-2)^3 - 13(-2) + 6$

$= -32 + 26 + 6 = 0$

(therefore $(x+2)$ is a factor).

(b) let $x = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right) + 6$

$= \frac{27}{2} - \frac{39}{2} + \frac{12}{2} = \frac{39}{2} - \frac{39}{2} = 0$

therefore $(2x-3)$ is a factor.

(c) $\dfrac{2x^2+x-6}{4x^3-13x+6}$ $\qquad 2x^2+x-6 = (2x-3)(x+2)$

$4x^3 - 13x + 6 = (x+2)(2x-3)(ax+b)$

equate coefficients of x^3 and constants.

$4 = 1 \times 2 \times a \Rightarrow \underline{\underline{a=2}}$

$6 = 2 \times -3 \times b \Rightarrow \underline{\underline{b=-1}}$

$$\frac{2x^2+x-6}{4x^3-13x+6} = \frac{\cancel{(2x-3)}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(2x-3)}(2x-1)} = \frac{1}{2x-1}$$

(2) When $t = 0 \quad P = 80$

a(i) so $A = 80$

(ii) $2000 = 80k^{25}$

$\frac{2000}{80} = k^{25} \Rightarrow 25 = k^{25}$

$\sqrt[25]{25} = k$

$k = 1.137411$

(b) $100000 = 80(1.137411)^t$

$1250 = 1.137411^t$

$\ln(1250) = t\ln(1.137411)$

$t = \frac{\ln(1250)}{\ln(1.137411)} = 55.38$

so 56 years $\Rightarrow$ 2016

(3)(a)(i) $(1-x)^{\frac{1}{3}} \approx 1 + \left(\frac{1}{3}\right)(-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(-x)^2$

$= 1 - \frac{1}{3}x - \frac{1}{9}x^2$

(ii) $(125-27x)^{\frac{1}{3}}$

$= \left[125\left(1-\frac{27}{125}x\right)\right]^{\frac{1}{3}} = 5\left(1-\frac{27}{125}x\right)^{\frac{1}{3}}$

$\approx 5\left[1 - \frac{1}{3}\left(\frac{27}{125}\right)x - \frac{1}{9}\left(\frac{27}{125}\right)^2 x^2\right]$

$= 5 - \frac{9}{25}x - \frac{81}{3125}x^2$

(b) $\sqrt[3]{119} = 119^{\frac{1}{3}}$ so let $125 - 27x = 119$

$\Rightarrow 27x = 6$

$x = \frac{6}{27} = \frac{2}{9}$

$5 - \frac{9}{25}\left(\frac{2}{9}\right) - \frac{81}{3125}\left(\frac{2}{9}\right)^2 = 4.91872$

(4) (a) $x = 3\cos 2\theta \qquad y = 2\cos\theta$

i) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\frac{dy}{d\theta} = -2\sin\theta \qquad \frac{dx}{d\theta} = -6\sin 2\theta \Rightarrow \frac{d\theta}{dx} = \frac{-1}{6\sin 2\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$ (double angle formula)

$\frac{d\theta}{dx} = \frac{-1}{6\sin 2\theta} = \frac{-1}{12\sin\theta\cos\theta}$

$\frac{dy}{dx} = -2\cancel{\sin\theta} \times \frac{-1}{12\cancel{\sin\theta}\cos\theta} = \frac{1}{6} \times \frac{1}{\cos\theta} = \frac{1}{6\cos\theta}$

ii) Gradient of tangent at $\theta = \frac{\pi}{3}$ is $\frac{1}{6\cos\frac{\pi}{3}} = \frac{1}{3}$

so gradient of normal at this point is $\boxed{-3}$

When $\theta = \frac{\pi}{3} \qquad x = 3\cos\left(\frac{2\pi}{3}\right) \qquad y = 2\cos\left(\frac{\pi}{3}\right)$

$= -\frac{3}{2} \qquad = \sqrt{3}$

$(y - \sqrt{3}) = +3\left(x + \frac{3}{2}\right)$

(b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 - \cos 2x \; dx$

(using double angle formula: $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$)

$\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2} - \frac{1}{2}\right] = \frac{\pi}{4} - \frac{1}{2}$

5 (a) $\vec{OA} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$ $\quad \vec{OB} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

$$s = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

b(i) $r = \begin{pmatrix} -8+5\mu \\ 5 \\ -6-2\mu \end{pmatrix}$ $\quad s = \begin{pmatrix} 5-\lambda \\ 1-2\lambda \\ -2+5\lambda \end{pmatrix}$

At point of intersection:

$5 = 1-2\lambda$

$2\lambda = -4$

$\boxed{\lambda = -2}$

$-8+5\mu = 5-\lambda$

$-8+5\mu = 7$

$5\mu = 15$

$\boxed{\mu = 3}$

Check 3rd equation:

$-6-2\mu = -6-6 = -12$

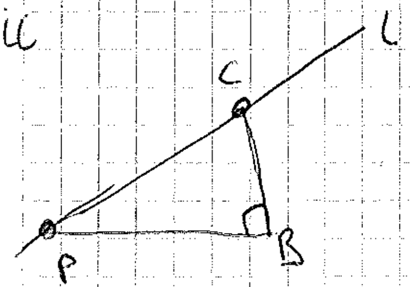
$-2+5\lambda = -2-10 = -12$ ✓ so lines intersect.

when $\lambda = -2 \quad \mu = 3$

$$\Rightarrow \begin{pmatrix} -8+5\mu \\ 5 \\ -6-2\mu \end{pmatrix} = \begin{pmatrix} -8+15 \\ 5 \\ -6-6 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -12 \end{pmatrix}$$

$P = (7, 5, -12)$

ii



$B(4, -1, 3)$

$P(7, 5, -12)$

$\vec{BP} \cdot \vec{CB} = 0$ (since perpendicular)

$$\vec{BP} = \vec{OP} - \vec{OB} = \begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix}$$

PTO.

Let $\vec{OC} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then $\vec{CB} = \vec{OB} - \vec{OC} = \begin{pmatrix} 4-a \\ -1-b \\ 3-c \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix} \cdot \begin{pmatrix} 4-a \\ -1-b \\ 3-c \end{pmatrix} = 0$$

C lies on L so $a = -8+5\mu \quad b = 5 \quad c = -6-2\mu$

$$\begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix} \cdot \begin{pmatrix} 4+8-5\mu \\ -6 \\ 3+6+2\mu \end{pmatrix} = 0$$

$$3(12-5\mu) + 6(-6) + (-15)(9+2\mu) = 0$$

$$\Rightarrow \cancel{36} - 15\mu - \cancel{36} - 135 - 30\mu = 0$$

$$45\mu = 135$$

$$\boxed{\mu = -3}$$

$$\vec{OC} = \begin{pmatrix} -8-15 \\ 5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} -23 \\ 5 \\ 0 \end{pmatrix}$$

C is point $(-23, 5, 0)$.

6 (a) $2y + e^{2x}y^2 = x^2 + c$

$x = 1 \quad y = \frac{1}{e}$

$\frac{2}{e} + e^2\left(\frac{1}{e}\right)^2 = 1 + c$

$\frac{2}{e} + 1 = 1 + c \quad \Rightarrow c = \frac{2}{e}$

(b) $\frac{dy}{dx}$

$\frac{d}{dx}2y + \frac{d}{dx}e^{2x}y^2 = \frac{d}{dx}x^2 + \frac{d}{dx}2e^{-1}$

$\frac{d}{dx}2y = \frac{d}{dy}2y \times \frac{dy}{dx} = 2\frac{dy}{dx}$

$\frac{d}{dx}e^{2x}y^2 = e^{2x}\frac{d}{dy}y^2 \times \frac{dy}{dx} + y^2\frac{d}{dx}e^{2x}$ (using product rule)

$= 2ye^{2x}\frac{dy}{dx} + 2y^2e^{2x}$

$\frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}2e^{-1} = 0$

$2\frac{dy}{dx} + 2ye^{2x}\frac{dy}{dx} + 2y^2e^{2x} = 2x$

$(2ye^{2x} + 2)\frac{dy}{dx} = 2x - 2y^2e^{2x}$

$\frac{dy}{dx} = \frac{2x - 2y^2e^{2x}}{2ye^{2x} + 2} = \frac{x - y^2e^{2x}}{ye^{2x} + 1}$

(6) Stationary point $\Rightarrow \dfrac{dy}{dx} = 0$

$\dfrac{x - y^2 e^{2x}}{y e^{2x} + 1}$ evaluated at $x = 1$, $y = \frac{1}{e} = e^{-1}$ gives:

$$\frac{1 - e^{-2}e^{2}}{e^{2}e^{-1} + 1} = \frac{1 - 1}{e + 1} = 0 \checkmark$$

(7) (a) $\dfrac{dA}{dt} = -k$

(b)(i) When $t = 0$ $\quad A = 4\pi \times 60^2 = 14400\pi$

$t = 9 \quad A = 4\pi \times 30^2 = 3600\pi$

Separate variables in (a)

$$\int dA = \int -k \, dt$$

$$\Rightarrow \boxed{A = -kt + C}$$

$14400\pi = 0 + C \quad \Rightarrow \quad C = 14400\pi$

$3600\pi = -9k + 14400\pi$

$9k = 10800\pi$

$k = 1200\pi$

$$\boxed{A = -1200\pi t + 14400\pi}$$

$A = -1200\pi(t - 12)$

$$\boxed{A = 1200\pi(12 - t)}$$

(ii) $A = 0 \Rightarrow \quad 0 = 1200\pi(12 - t)$

$\Rightarrow t = 12$ $\qquad$ 12 days

8) $$\frac{1}{(3-2x)(1-x)^2} = \frac{A}{(3-2x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

$\times (3-2x)(1-x)^2$

$$1 = A(1-x)^2 + B(3-2x)(1-x) + C(3-2x)$$

let $\underline{x=1}$ $\Rightarrow$ $1 = 0 + 0 + C(3-2)$

$\Rightarrow 1 = C$

$\boxed{C=1}$

let $\underline{x=\frac{3}{2}}$ $\Rightarrow$ $1 = A\left(\frac{-1}{2}\right)^2 + 0 + 0$

$1 = \frac{A}{4}$

$\boxed{A=4}$

To find B you can now use any value for x, so use $x=0$.

$1 = 4(1)^2 + B(3)(1) + 3$

$1 = 7 + 3B$

$3B = -6 \Rightarrow \boxed{B=-2}$

$$\frac{1}{(3-2x)(1-x)^2} = \frac{4}{(3-2x)} - \frac{2}{(1-x)} + \frac{1}{(1-x)^2}$$

(b) $\frac{dy}{dx} = \frac{2\sqrt{y}}{(3-2x)(1-x)^2}$ Separating variables gives

$$\int \frac{1}{2\sqrt{y}}\, dy = \int \frac{1}{(3-2x)(1-x)^2}\, dx$$

$$\frac{1}{2}\int y^{-\frac{1}{2}}\, dy = \int 4(3-2x)^{-1} - 2(1-x)^{-1} + (1-x)^{-2}\, dx \quad \text{(using (a))}$$

$$\frac{1}{2}\left[2y^{\frac{1}{2}}\right] = 4 \times -\frac{1}{2}\ln(3-2x) + 2\ln(1-x) + \frac{1}{1-x} + C$$

$$\sqrt{y} = -2\ln(3-2x) + 2\ln(1-x) + \frac{1}{1-x} + C$$

use fact that $y=0$ when $x=0$ to find C.

$$0 = -2\ln 3 + 1 + C$$

$$C = 2\ln 3 - 1$$

so $\boxed{\sqrt{y} = -2\ln(3-2x) + 2\ln(1-x) + \frac{1}{1-x} + 2\ln 3 - 1}$

Simplify RHS:

consider first $\frac{1}{1-x} - 1 = \frac{1}{1-x} - \frac{1-x}{1-x} = \frac{1-1+x}{1-x} = \frac{x}{1-x}$

Then:

$$-2\ln(3-2x) + 2\ln(1-x) + 2\ln 3$$

$$= \ln(3-2x)^{-2} + \ln(1-x)^2 + \ln 3^2$$

$$= \ln\left[\frac{3^2(1-x)^2}{(3-2x)^2}\right] = \ln\left[\frac{3(1-x)}{(3-2x)}\right]^2 = 2\ln\left[\frac{3-3x}{3-2x}\right]$$

$$\Rightarrow \boxed{y^{\frac{1}{2}} = 2\ln\left[\frac{3-3x}{3-2x}\right] + \frac{x}{1-x}}$$